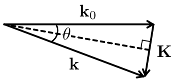
Problem set 5 – Soft Matter

Problem 19

a) Show that magnitude of the scattering vector is $K \equiv \mid \vec{K} \mid = \frac{4\pi}{\lambda} \sin\left(\frac{\theta}{2}\right)$ using the figure below.



$|\mathbf{k}| = |\mathbf{k}_0| = k$

The form factor $P(K)$ for a homogeneous (i.e. $\zeta(\mathbf{r}) = \zeta$) sphere of radius a – in spherical coordinates – is

$$P(K) = \left| \frac{\int_0^a \frac{\sin(Kr)}{Kr} 4\pi r^2 dr}{\int_0^a 4\pi r^2 dr} \right|^2 . \qquad \text{(Eq. 1)}$$

b) Show that the form factor for homogeneous spherical particles is given by

$$P(K) = \left(3 \, \frac{\sin(Ka) - Ka\cos(Ka)}{(Ka)^3} \right)^2 .$$

Hint: make the variable substitution $x = Kr$ and then integrate by parts $\int f'g \, dx = [fg] - \int fg' dx$.

c) The first minimum of $P(K)$ is located at $Ka = 4.49$ (which follows from the numerical solutions of $\tan x = x$). Calculate the scattering angle corresponding to this first minimum for a sphere of $a = 1$ μm, and $\lambda = 532$ nm. Does this scattering angle increase or decrease for smaller particles?

d) Explain why the minima in the form factor become less sharp for polydisperse particles.

Problem 20

The position autocorrelation function for a Brownian particle in a harmonic optical trap is given by

$$\langle x(t')x(t) \rangle = \frac{k_B T}{\kappa} e^{-|t'-t|/\tau},$$

where $\tau = \xi/\kappa$ is the relaxation time of the particle in the trap, ξ the drag coefficient and κ the (harmonic) trap stiffness.

a) Show that the mean squared displacement (MSD), defined as $\langle [x(t') - x(t)]^2 \rangle$, is given by

$$\langle [x(t') - x(t)]^2 \rangle = \frac{2k_B T}{\kappa} \left(1 - e^{-\Delta t/\tau} \right),$$

with $\Delta t = |t' - t|$ the lag time (the MSD depends on the time difference, not on the absolute time).

b) Determine the following two limits for the MSD:

   i) For $\Delta t \ll \tau \quad \rightarrow \quad \text{MSD}(\Delta t) = \frac{2k_B T}{\xi} \Delta t.$

   ii) For $\Delta t \gg \tau \quad \rightarrow \quad \text{MSD}(\Delta t) = \frac{2k_B T}{\kappa}.$

c) Sketch the MSD as a function of the lag time (Δt) and also indicate the limits from part b).

d) For a 1.0 μm radius polystyrene particle in water ($\eta = 0.98$ mPa s) at $T = 298$ K in an optical trap with a stiffness of $\kappa = 34.95$ fN/nm, the long-time plateau value of the MSD is 231 nm^2 and the initial slope $\approx 400 \cdot 10^{-15}$ m^2/s. Determine Boltzmann's constant (k_B), and hence Avogadro's number, and discuss how else you could determine k_B from a measurement of the MSD of a particle in an optical trap (you could try it, which one is easier/works better?).

Problem 21

For small scattering vectors, the form factor $P(K)$ is well-described by Guinier's law

$$P(K) = \exp\left(-\frac{1}{3}K^2R_G^2\right), \qquad \text{with} \qquad R_G^2 = \frac{\int_0^a r^4 dr}{\int_0^a r^2 dr},$$

the radius of gyration (as seen in lecture 3 on polymers). In this problem we will derive Guinier's law.

a) Starting from the definition of the form factor (Eq. 1 from problem 19) and given that $\frac{\sin Kr}{Kr} \approx 1 - \frac{1}{6}K^2r^2 + \ldots$, show that $P(K)$ can be written as

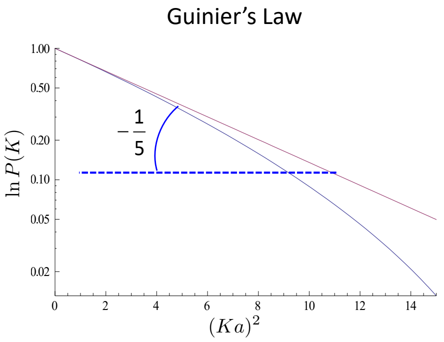
$$P(K) = 1 - \frac{1}{3}K^2\frac{\int_0^a r^4 dr}{\int_0^a r^2 dr} + \mathcal{O}(k^4)$$

b) Neglecting the higher-order terms ($\mathcal{O}(k^4)$), show that for small K Guinier's law is obtained

$$P(K) = 1 - \frac{1}{3}K^2R_G^2 = \exp\left(-\frac{1}{3}K^2R_g^2\right).$$

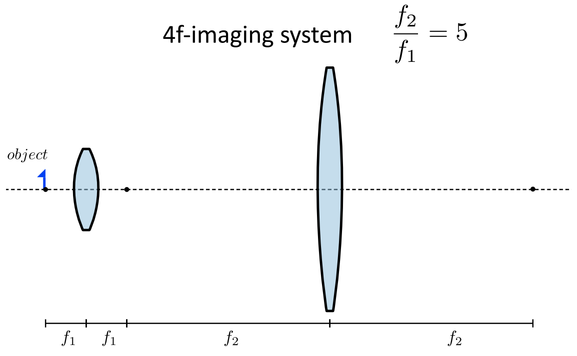
c) Calculate the radius of gyration for a homogeneous sphere of radius a.

d) Using your answer to part (c), verify that a plot of $\ln(P(K))$ vs $(Ka)^2$ has a slope of $-1/5$ in the limit of small K as shown in the figure below.

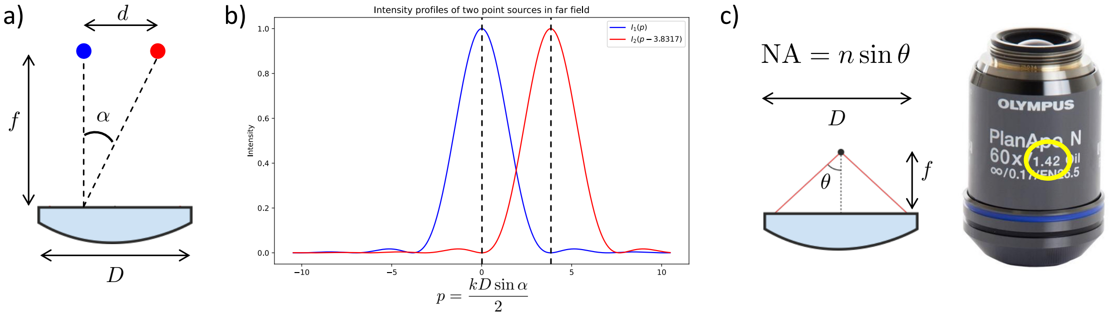


Problem 22

a) The figure below shows a $4f$–imaging system with a magnification $M = f_2/f_1 = 5$. Draw the ray tracing and construct the image of the object.



Next, we will 'derive' Abbe's diffraction limit, which sets the resolution of an optical microscope. To this end, we will consider the intensity profiles $I(\theta)$ of two emitting point sources separated by a distance d at a focal distance f away from the imaging lens (see panel a) in figure below)). These point sources can be resolved if the (minimum) distance between them corresponds to the intensity maximum of the first point source overlapping with the first intensity minimum of the second one, as shown in panel b) of the figure below). For this situation, $p_{min} = \frac{1}{2}kD\sin\alpha = 3.8317$, where D is the diameter of the imaging lens, $k = 2\pi/\lambda$ (wavevector) and α the angle between the point sources (see panel a)).



Panel a) Two point sources separated by a distance d in the focal plane of the lens.
Panel b) The intensity profiles of two point sources separated by $p_{min} = 3.8317$.
Panel c) The geometry for the numerical aperture for an objective lens.

b) Show that the diffraction limit of the lens is given by $d = 1.22 f\lambda/D$. Note that $f \gg d$.

c) Next, we relate the diffraction limit to the numerical aperture (NA) of a lens: $\mathrm{NA} = n\sin\theta$, with n the refractive index of the medium and θ is the collection angle of the objective (see panel c) in the figure above). Using that $f \gg D$ and $n = 1$, show that the diffraction limit can be written as

$$d = \frac{1.22\lambda}{2\ \mathrm{NA}} \approx \frac{\lambda}{2\ \mathrm{NA}} \qquad \text{(as it is often seen in textbooks)}$$

Note that the assumption $f \gg D$ is actually not necessarily true for microscopes.

d) Calculate the diffraction limit for a microscope objective with NA=1.42 and a wavelength of 532 nm.